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Three-dimensional problems of unsteady diffusion boundary layer

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Abstract—In the diffusion boundary layer approximation an exact analytical solution is obtained for the problem of unsteady convective mass exchange between a spherical droplet (bubble) and an arbitrary three-dimensional linear shear flow, with the unperturbed velocity field assigned by the symmetric shear tensor. The dependence of Sherwood number on time and Peclet number is established. A simple approximate formula is presented for calculating the rate of unsteady mass exchange of droplets and solid particles with an arbitrary steady flow. The stationary problem of mass exchange between a droplet and a linear shear flow in the presence of the first-order volumetric chemical reaction is considered. An equation is suggested to compute the Sherwood number for a droplet or a particle of arbitrary shape and for any type of flow at large Peclet numbers over the entire range of reaction rate constants.

1. INTRODUCTION

UNSTEADY diffusion to a spherical droplet (bubble) in a developed translational Stokes flow at large Peclet numbers was considered in refs. [1, 2]. A similar problem of convective mass and heat transfer for potential flow of an ideal fluid about a droplet was studied in refs. [2, 3]. In ref. [4] (see also ref. [5]) the solution was obtained to the problem of unsteady mass transfer of a spherical droplet in an axisymmetric shear Stokes flow.

Diffusion to a droplet in a translational Stokes flow in the presence of a first-order volumetric chemical reaction was examined in ref. [6]. The problems of stationary mass transfer of a solid spherical particle [7] and a droplet [8] in three-dimensional translational flow were investigated. A comprehensive treatment of the results and methods of solution of corresponding problems, as well as a sufficiently complete bibliography on the subject can be found in ref. [5].

The present study is the first to provide examples which illustrate exact integration of three-dimensional non-stationary equations for a diffusion boundary layer.

2. FORMULATION OF THE PROBLEM. THREE-DIMENSIONAL ANALOGUE OF THE STREAM FUNCTION

Consider unsteady convective diffusion to the surface of a droplet (bubble) of arbitrary shape in a laminar viscous incompressible fluid flow. Assume that the main resistance to transfer is concentrated in the continuous phase. The method given below is also appropriate in the case of convective mass and heat exchange between a particle and an ideal fluid.

Suppose that at the initial instant of time $t_* = 0$ the concentration beyond the droplet is the same and equal to C_0, whereas when $t_* > 0$, on the droplet surface there occurs a complete absorption of the substance dissolved in the fluid (i.e. the diffusional regime of reaction is realized which corresponds to zero concentration on the droplet surface). It will be considered that the Peclet number $Pe = aU/D$ is large as usual (here a is the characteristic size of the droplet (for example, radius), U the characteristic flow velocity and D the diffusion coefficient). For simplicity, the flow field is assumed to be established and known from the solution of the corresponding hydrodynamic problem.

To analyse the diffusion boundary layer, the local orthogonal curvilinear system of dimensionless coordinates ξ, η, λ will be introduced which is fixed with respect to the droplet surface and streamlines [5, 8]. Suppose that the vector of the normal to the droplet surface determines the direction of the unit vector $\mathbf{e}_\xi$ (see Fig. 1). The direction of the fluid velocity vector at this point of the surface prescribes the direction of the unit vector $\mathbf{e}_\eta$. The unit vector $\mathbf{e}_\lambda$ is perpendicular to both $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$. In such a coordinate system the presentation $\mathbf{v} = \{v_\xi, v_\eta, 0\}$ is valid for the fluid velocity vector. To be specific, it is further assumed that the droplet surface is prescribed by the fixed value $\xi = \xi_s$.

Taking into account the fact that in the coordinate system chosen the continuity equation for incompressible fluid has the form

$$\operatorname{div} \mathbf{v} = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial \xi}\left(v_\xi \sqrt{\left(\frac{g}{g_{\xi\xi}}\right)}\right) + \frac{\partial}{\partial \eta}\left(v_\eta \sqrt{\left(\frac{g}{g_{\eta\eta}}\right)}\right)\right] = 0, \quad g = g_{\xi\xi} g_{\eta\eta} g_{\lambda\lambda} \quad (1)$$

where $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\lambda\lambda}$ are the metric tensor components.

NOMENCLATURE

- a — characteristic size of droplet, bubble or solid particle (radius of a spherical particle)
- C_* — concentration in flow
- C_0 — concentration at the initial time instant
- C_s — concentration on the surface of droplet (particle, bubble)
- c — dimensionless concentration in non-stationary problems, $(C_0 - C_*)/C_0$
- $\bar{c}$ — dimensionless concentration in stationary problems with volumetric reaction, C_*/C_s
- D — diffusion coefficient
- E^*_{ij}, E_{ij} — dimensional and dimensionless shear tensor components
- E^*_i, E_i — dimensional and dimensionless shear tensor components in the Cartesian coordinate system fixed with respect to the principal axes of shear tensors
- f — function determining the stream function analogue, equation (6)
- $g_{\xi\xi}$, $g_{\eta\eta}$, $g_{\lambda\lambda}$ — metric tensor components
- g — third metric tensor invariant, $g_{\xi\xi}\, g_{\eta\eta}\, g_{\lambda\lambda}$
- I — dimensionless integral (total) diffusional flux onto droplet surface
- J^*_2 — $(E^*_{ij}E^*_{ij})^{1/2} = (E^{*2}_1 + E^{*2}_2 + E^{*2}_3)^{1/2}$
- J_i — shear tensor invariants, equation (36)
- j — dimensionless diffusional flux
- K — first-order volumetric chemical reaction rate constant
- k — dimensionless volumetric chemical reaction rate constant, a^2K/D
- Pe — Peclet number, aU/D
- r, θ, ϕ — spherical coordinate system with origin fixed at droplet centre
- Sh — mean Sherwood number in non-stationary problems
- $\overline{Sh}$ — mean Sherwood number in stationary problems with volumetric reaction
- Sh_∞ — mean Sherwood number corresponding to steady mass exchange regime, $\lim_{t\to\infty} Sh$
- $\overline{Sh}_0$ — mean Sherwood number in the absence of volumetric reaction, $\lim_{k\to 0} \overline{Sh}$
- t_* — time
- t — dimensionless time, Ut_*/a
- U — characteristic flow velocity
- U_∞ — unperturbed translational flow velocity far from droplet
- $\mathbf{v}$ — fluid velocity vector
- v_θ, v_ϕ — tangential components of fluid velocity vector on droplet surface
- x_i — Cartesian coordinate system with origin fixed at droplet centre ($i = 1, 2, 3$)
- z — time-like variable determined by equation (18), $z(t, \eta, \lambda)$.

Greek symbols

- β — ratio of dynamic velocities of droplet and surrounding fluid ($\beta = 0$ corresponds to a gas bubble)
- ξ, η, λ — curvilinear coordinate system fixed with respect to droplet surface and streamlines
- τ — dimensionless time, $t/Pe = Dt_*/a^2$
- ψ — three-dimensional analogue of function (in plane and axisymmetric case is identical with conventional stream function).

Superscript and subscript

- s — quantities are taken on droplet surface.

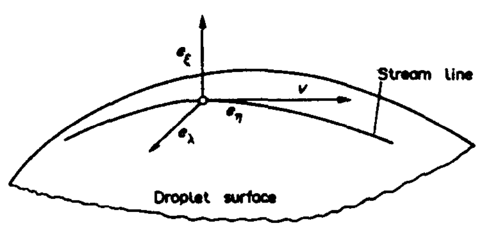

FIG. 1. Curvilinear orthogonal coordinate system fixed with respect to the droplet surface and streamlines.

The function $\psi(\xi, \eta, \lambda)$ will be defined as the solution of the system

$$\frac{\partial\psi}{\partial\xi} = v_\eta\sqrt{\left(\frac{g}{g_{\eta\eta}}\right)}, \quad \frac{\partial\psi}{\partial\eta} = -v_\xi\sqrt{\left(\frac{g}{g_{\xi\xi}}\right)}. \qquad (2)$$

Then, continuity equation (1) which agrees with the requirement of the system integrability, is satisfied automatically. The integration constant is selected so that the function ψ would transform into zero on the droplet surface at $\xi = \xi_s$.

The surfaces $\psi(\xi, \eta, \lambda) = \text{const.}$ are composed entirely of streamlines. The function ψ is simple physically, i.e. it is a three-dimensional analogue of the stream function. In the plane and axisymmetric cases it coincides with the conventional stream function.

In dimensionless variables of the curvilinear coordinate system ξ, η, λ the equation of the unsteady-state diffusion boundary layer and also the corresponding initial and boundary conditions have the form

$$\frac{\partial c}{\partial t} + \frac{1}{\sqrt{g^s}}\left(\frac{\partial\psi}{\partial\xi}\frac{\partial c}{\partial\eta} - \frac{\partial\psi}{\partial\eta}\frac{\partial c}{\partial\xi}\right) = \frac{1}{Pe}\frac{1}{g^s_{\xi\xi}}\frac{\partial^2 c}{\partial\xi^2} \qquad (3)$$

$$t = 0, c = 0, \quad \xi = \xi_s, c = 1; \quad \xi \to \infty, c \to 0 \quad (4)$$

$$c = \frac{C_0 - C_*}{C_0}, \quad t = \frac{Ut_*}{a}, \quad Pe = \frac{aU}{D}. \quad (5)$$

Here and hereafter the superscript 's' means that the corresponding quantities are taken on the droplet surface at $\xi = \xi_s$; g^s and $g^s_{\xi\xi}$ are known functions depending only on two curvilinear coordinates η and λ; C_* is the concentration in the flow.

As usual, in deriving equation (3) the diffusion transfer along the droplet surface was neglected in comparison with that along the normal, and only higher order terms in the expansion of the metric tensor components were taken into account when $\xi \to \xi_s$; moreover, use was made of the relationship between the fluid velocity vector components and stream function analogue (2).

Allowing for the condition of the droplet surface impermeability to the fluid, the fluid velocity components with $\xi \to \xi_s$ will be defined as

$$v_\xi = (\xi - \xi_s)O(1), \quad v_\eta = O(1).$$

Therefore, due to equalities (2), the following representation is valid for the stream function analogue in the boundary layer:

$$\psi = (\xi - \xi_s)f(\eta, \lambda). \quad (6)$$

In deriving equation (6), the following relations were taken account of:

$$g_{\xi\xi} = O(1), \quad g_{\eta\eta} = O(1), \quad g_{\lambda\lambda} = O(1)$$

which were valid near the droplet surface when $\xi \to \xi_s$.

For simplicity it is assumed below that in the flow region under consideration the inequality $f \geqslant 0$ is satisfied when $\xi \geqslant \xi_s$.

It should be noted that the third curvilinear coordinate enters into equation (3) only parametrically, therefore the dependence of the functions employed in the study on λ is not further indicated.

3. THE METHOD OF SOLUTION

In the diffusion boundary layer equation (3) the change-over will be made from the variable ξ to the new variable ζ, according to the equation

$$\zeta = \psi\sqrt{(Pe)} \quad (7)$$

where the function ψ is determined by equation (6). This will yield

$$\frac{\partial c}{\partial t} + \frac{f}{\sqrt{g^s}}\frac{\partial c}{\partial \eta} = \frac{f^2}{g^s_{\xi\xi}}\frac{\partial^2 c}{\partial \zeta^2}. \quad (8)$$

The distribution of concentration is sought in the form

$$c = c(\zeta, z) \quad (9)$$

where $z = z(t, \eta, \lambda)$ is the new time-like variable found below in the course of problem solution. Substituting equation (9) into equation (8) one obtains

$$\left(\frac{\partial z}{\partial t} + \frac{f}{\sqrt{g^s}}\frac{\partial z}{\partial \eta}\right)\frac{\partial c}{\partial z} = \frac{f^2}{g^s_{\xi\xi}}\frac{\partial^2 c}{\partial \zeta^2}. \quad (10)$$

Now, let the requirement be imposed that the function z should satisfy the following first-order linear partial differential equation:

$$\frac{\partial z}{\partial t} + \frac{f}{\sqrt{g^s}}\frac{\partial z}{\partial \eta} = \frac{f^2}{g^s_{\xi\xi}} \quad (11)$$

as well as the additional condition that

$$t = 0, \quad z = 0. \quad (12)$$

Then equation (10) is reduced to the standard heat conduction equation

$$\frac{\partial c}{\partial z} = \frac{\partial^2 c}{\partial \zeta^2}. \quad (13)$$

The initial and boundary conditions for equation (13) follow from equation (4), allowing for equations (6), (7) and (12), and have the form

$$z = 0, c = 0; \quad \zeta = 0, c = 1; \quad \zeta \to \infty, c \to 0. \quad (14)$$

The solution of problem (13), (14) is well known

$$c = \operatorname{Erfc}\left(\frac{\zeta}{2\sqrt{z}}\right),$$

$$\operatorname{Erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-x^2)\,dx. \quad (15)$$

The relation $z = z(t, \eta)$ will now be found from equation (11) which is equivalent to the following system of ordinary differential equations:

$$dt = \frac{\sqrt{g^s}}{f}\,d\eta = \frac{g^s_{\xi\xi}}{f^2}\,dz. \quad (16)$$

Integrating the first two and the last two equations (16) obtain the first integrals

$$t - \int \frac{\sqrt{g^s}}{f}\,d\eta = A_1, \quad z - \int \frac{\sqrt{g^s}}{g^s_{\xi\xi}} f\,d\eta = A_2 \quad (17)$$

where A_1 and A_2 are arbitrary constants. Therefore, the general solution to equation (11) has the form (Φ is an arbitrary function)

$$z = \int \frac{\sqrt{g^s}}{g^s_{\xi\xi}} f\,d\eta + \Phi\left(t - \int \frac{\sqrt{g^s}}{f}\,d\eta\right). \quad (18)$$

The explicit form of the function Φ is ascertained with the aid of condition (12). As a result, the following expression will be obtained for the variable z:

$$z = \int_{T(\omega)}^{\eta} \frac{\sqrt{g^s}}{g^s_{\xi\xi}} f\,d\eta, \quad \omega = t - \int \frac{\sqrt{g^s}}{f}\,d\eta \quad (19)$$

where the function $T = T(\omega)$ is found from the equality

$$T\left(-\int \frac{\sqrt{g^s}}{f}\,d\eta\right) = \eta.$$

The use of equations (6), (7) and (15) for the dimensionless local diffusion flux on the droplet surface gives

$$j = -\frac{1}{\sqrt{g^s_{\xi\xi}}}\left(\frac{\partial c}{\partial \xi}\right)_{\xi=\xi_s} = \frac{f(\eta,\lambda)\sqrt{(Pe)}}{\sqrt{(\pi g^s_{\xi\xi}(\eta,\lambda) z(t,\eta,\lambda))}}. \tag{20}$$

For the dimensionless total diffusion flux on the droplet surface $S = \{\xi = \xi_s,\ 0 \leqslant \eta \leqslant H,\ 0 \leqslant \lambda \leqslant \Lambda\}$ one obtains

$$I = \iint_S j\,dS = \int_0^\Lambda \int_0^H j\sqrt{g^s_{\eta\eta}}\sqrt{g^s_{\lambda\lambda}}\,d\eta\,d\lambda$$

$$= \frac{\sqrt{(Pe)}}{\sqrt{\pi}} \int_0^\Lambda \int_0^H \frac{f\sqrt{g^s}\,d\eta\,d\lambda}{g^s_{\xi\xi}\sqrt{z}}. \tag{21a}$$

In the two-dimensional case ($\partial/\partial\lambda = 0$) equation (21a) simplifies to

$$I = \Lambda\frac{\sqrt{(Pe)}}{\sqrt{\pi}} \int_0^H \frac{f\sqrt{g^s}\,d\eta}{g^s_{\xi\xi}\sqrt{z}}. \tag{21b}$$

Here the value $\Lambda = 2\pi$ corresponds to the axisymmetric case and $\Lambda = 1$ to the plane case.

4. FORMULAE FOR COMPUTING NEW VARIABLES AND DIFFUSION FLUXES IN THE SPHERICAL COORDINATE SYSTEM

Usually the initial information on the flow field enables one to directly obtain only the distribution of fluid velocities near the droplet in a certain orthogonal fixed coordinate system ξ, θ, ϕ fixed only with respect to its surface (and not to streamlines). Therefore, in the general case of a three-dimensional flow field it is first necessary to solve the auxiliary problem of determining the curvilinear coordinate system ξ, η, λ described previously and to find the expansion of the stream function analogue near the droplet surface (6); after that it is possible to use equations (15), (20) and (21a) to calculate the basic characteristics of unsteady-state mass transfer.

It will be shown in which way the diffusion fluxes and the functions f and z are calculated for a droplet (bubble) of spherical shape in the three-dimensional case. It will be assumed that the tangential components of the fluid velocity vector near the droplet surface are known

$$v_\theta = v_\theta(\theta,\phi), \quad v_\phi = v_\phi(\theta,\phi) \quad (r \to 1). \tag{22}$$

Here the dimensionless spherical system of coordinates r, θ, ϕ is used with the origin fixed at the droplet centre. The coordinate ξ and the metric tensor components (at $\xi = \xi_s$) are given by the formulae

$$\xi = r-1;\quad g^s_{\xi\xi} = 1,\ g^s_{\theta\theta} = 1,\ g^s_{\phi\phi} = \sin^2\theta. \tag{23}$$

The sought orthogonal curvilinear coordinates $\eta = \eta(\theta,\phi)$ and $\lambda = \lambda(\theta,\phi)$ should by definition satisfy the relations

$$\mathbf{v}_\tau = \text{const.}\,\nabla\eta,\quad (\mathbf{v}_\tau\cdot\nabla\lambda) = 0$$

$$\mathbf{v}_\tau = \mathbf{e}_\theta v_\theta + \mathbf{e}_\phi v_\phi,\quad \nabla = \mathbf{e}_\theta\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}.$$

Hence, there are the following first-order partial differential equations for η and λ:

$$v_\phi\frac{\partial\eta}{\partial\theta} - \frac{v_\theta}{\sin\theta}\frac{\partial\eta}{\partial\phi} = 0 \tag{24}$$

$$v_\theta\frac{\partial\lambda}{\partial\theta} + \frac{v_\phi}{\sin\theta}\frac{\partial\lambda}{\partial\phi} = 0. \tag{25}$$

The dependence of the sought curvilinear coordinates η and λ on the spherical coordinates θ and ϕ will be determined by the first integrals of the characteristic equations

$$v_\theta\,d\theta = -\sin\theta v_\phi\,d\phi \tag{26}$$

$$v_\phi\,d\theta = \sin\theta v_\theta\,d\phi \tag{27}$$

which comply with equations (24) and (25).

Assuming now the functions $\eta = \eta(\theta,\phi)$ and $\lambda = \lambda(\theta,\phi)$ to be known, use will be made of the fact that the square of the length on the droplet surface is preserved during transition from the old, θ, ϕ, to the new, η, λ, coordinate system

$$d\theta^2 + \sin^2\theta\,d\phi^2 = g^s_{\eta\eta}\,d\eta^2 + g^s_{\lambda\lambda}\,d\lambda^2$$

$$\left(d\eta = \frac{\partial\eta}{\partial\theta}d\theta + \frac{\partial\eta}{\partial\phi}d\phi,\quad d\lambda = \frac{\partial\lambda}{\partial\theta}d\theta + \frac{\partial\lambda}{\partial\phi}d\phi\right). \tag{28}$$

Taking account of equations (24) and (25) and making some transformations, obtain from equation (28) the sought expressions for the metric coefficients $g_{\eta\eta}$ and $g_{\lambda\lambda}$

$$g^s_{\eta\eta} = \frac{v_\theta^2}{v^2}\left(\frac{\partial\eta}{\partial\theta}\right)^{-2} = \sin^2\theta\frac{v_\phi^2}{v^2}\left(\frac{\partial\eta}{\partial\phi}\right)^{-2}$$

$$v^2 = v_\theta^2 + v_\phi^2$$

$$g^s_{\lambda\lambda} = \frac{v_\phi^2}{v^2}\left(\frac{\partial\lambda}{\partial\theta}\right)^{-2} = \sin^2\theta\frac{v_\theta^2}{v^2}\left(\frac{\partial\lambda}{\partial\phi}\right)^{-2}. \tag{29}$$

Now substitute equation (6) into the first equation (2) on the right-hand side of which only the main expansion terms were retained for $\xi \to \xi_s$. Further, allowing for the equality $v_\eta = |\mathbf{v}^s|$, the function f will be found which specifies the stream function analogue

$$f = v\frac{\sqrt{g^s}}{\sqrt{g^s_{\xi\xi}}},\quad v = v_\eta = \sqrt{(v_\theta^2 + v_\phi^2)}. \tag{30}$$

In the integrals, which determine the time-like variable z, equation (18), and which appear in equations (17)–(21a), the change-over should be made from the integration variable η to the spherical coordinate θ (or

ϕ) using the following relations, which are valid at $\lambda = \text{const.}$:

$$\sqrt{g^s_{\eta\eta}}\,\mathrm{d}\eta = \frac{v}{v_\theta}\,\mathrm{d}\theta = \sin\theta\frac{v}{v_\phi}\,\mathrm{d}\phi. \qquad (31)$$

These relations are derived from equation (28) taking into account the fact that the equality $\lambda = \text{const.}$ is fulfilled on the integral curves of the characteristic ordinary differential equation (27) which conforms to the partial differential equation (25).

Using equations (29)–(31) for the integrals in equation (18), obtain

$$\int^\eta \frac{\sqrt{g^s}}{f}\,\mathrm{d}\eta = \int^\eta \frac{\sqrt{g^s_{\eta\eta}}}{v}\,\mathrm{d}\eta = \int^\theta \frac{\mathrm{d}\theta}{\{v_\theta\}_\lambda} = \int^\phi \left\{\frac{\sin\theta}{v_\phi}\right\}_\lambda \mathrm{d}\phi$$

$$\int^\eta \frac{\sqrt{g^s}}{g^s_{\xi\xi}} f\,\mathrm{d}\eta = \int^\eta v g^s_{\lambda\lambda}\sqrt{g^s_{\eta\eta}}\,\mathrm{d}\eta = \int^\theta \sin^2\theta$$

$$\left\{v_\theta\left(\frac{\partial\lambda}{\partial\phi}\right)^{-2}\right\}_\lambda \mathrm{d}\theta = \int^\phi \left\{\sin\theta v_\phi\left(\frac{\partial\lambda}{\partial\theta}\right)^{-2}\right\}_\lambda \mathrm{d}\phi. \qquad (32)$$

Here, $\{F\}_\lambda\,\mathrm{d}\phi$ (or $\{F\}_\lambda\,\mathrm{d}\theta$) in the integrands means that the function $F(\theta, \phi)$ is written in terms of the variables λ and ϕ (or λ and θ) with the aid of the relation $\lambda = \lambda(\theta, \phi)$ and in integration λ is considered to be a parameter.

The Jacobian corresponding to the transition from the coordinates η, λ to θ, ϕ is calculated from the equation

$$\frac{\partial(\eta,\lambda)}{\partial(\theta,\phi)} = \frac{1}{\sqrt{g^s_{\eta\eta}}}\frac{v}{v_\theta}\frac{\partial\lambda}{\partial\phi} = -\frac{1}{\sqrt{g^s_{\eta\eta}}}\frac{v}{v_\phi}\frac{\partial\lambda}{\partial\theta}.$$

Transforming in integrand (21a) to the spherical coordinates θ, ϕ allowing for equations (25), (29), (30), the following equation will be obtained for computing the integral diffusion flux onto the droplet surface:

$$\begin{aligned} I &= \frac{\sqrt{(Pe)}}{\sqrt{\pi}}\int_0^{2\pi}\int_0^\pi \frac{f\sqrt{g^s}}{g^s_{\xi\xi}\sqrt{z}}\left|\frac{\partial(\lambda,\eta)}{\partial(\theta,\phi)}\right|\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{\sqrt{(Pe)}}{\sqrt{\pi}}\int_0^{2\pi}\int_0^\pi \frac{v}{\sqrt{z}} g^s_{\lambda\lambda}\sqrt{g^s_{\eta\eta}}\left|\frac{\partial(\eta,\lambda)}{\partial(\theta,\phi)}\right|\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{\sqrt{(Pe)}}{\sqrt{\pi}}\int_0^{2\pi}\int_0^\pi \frac{\sin^2\theta|v_\theta|}{\sqrt{z}}\left|\frac{\partial\lambda}{\partial\phi}\right|^{-1}\mathrm{d}\theta\,\mathrm{d}\phi \\ &= \frac{\sqrt{(Pe)}}{\sqrt{\pi}}\int_0^{2\pi}\int_0^\pi \frac{\sin\theta|v_\phi|}{\sqrt{z}}\left|\frac{\partial\lambda}{\partial\theta}\right|^{-1}\mathrm{d}\theta\,\mathrm{d}\phi. \end{aligned} \qquad (33a)$$

Thus, the calculation of the most significant mass transfer, i.e. of the integral flux (or of the mean Sherwood number) is performed in four steps. First, the components of fluid velocity on the droplet surface are determined, equation (22). Further, the general solution is found for characteristic equation (27), the substitution of the arbitrary integration constant of which by λ yields the relation $\lambda = \lambda(\theta, \phi)$. As the third step, the variable z, equation (18), is calculated with account being taken of equations (19) and (32) (recall that the integrands in equation (32) should be first written only in terms of the coordinate λ and that the variable over which the integration is performed). Finally, direct evaluation of the diffusion flux is made with the aid of any of the double integrals from equation (33a). In the axisymmetric case ($\partial/\partial\lambda = 0$) the diffusion flux should be calculated with the use of

$$\begin{aligned} I &= 2\sqrt{(\pi\,Pe)}\int_0^\pi \frac{\sin\theta\,f(\theta)\,\mathrm{d}\theta}{\sqrt{(z(t,\theta))}} \\ &= 2\sqrt{(\pi\,Pe)}\int_0^\pi \frac{\sin^2\theta|v_\theta|\,\mathrm{d}\theta}{\sqrt{(z(t,\theta))}} \end{aligned} \qquad (33b)$$

which were derived allowing for the equality $v_\phi = 0$ and for equations (21b), (23) and (30) at $\eta = 0$.

Now it will be shown in which way the aforegoing results are applied for solving specific problems of a non-stationary three-dimensional diffusion boundary layer.

5. UNSTEADY MASS TRANSFER TO A SPHERICAL DROPLET IN AN ARBITRARY DEFORMATIONAL SHEAR FLOW

Consider unsteady diffusion to a spherical droplet (bubble) in an arbitrary developed linear deformational shear flow the unperturbed velocity field of which at infinity has the form

$$r \to \infty,\ v_i = E_{ij}x_j + o(1);$$
$$E_{ij} = E_{ji} \quad (E_{11}+E_{22}+E_{33} = 0) \qquad (34)$$

where v_i and E_{ij} are the dimensionless components of velocity and shear tensor (the normalization of which will be shown further) written in terms of the Cartesian coordinate system with the origin fixed at the droplet centre; here and hereafter the summation is performed over the repeated indices i and j ($i, j = 1, 2, 3$). The fact that the tensor diagonal elements are equal to zero results from the condition of fluid incompressibility $\operatorname{div}\mathbf{v} = 0$; the symmetry of the shear tensor components under the permutation of indices i and j corresponds to the absence of the rotational component of fluid velocity far from the drop.

In the Stokes approximation the solution for the problem of the developed deformational shear flow around a spherical droplet, equation (34) is given by the expression [9]

$$v_i = E_{ij}x_j\left(1-\frac{\beta}{\beta+1}\frac{1}{r^5}\right) - E_{jk}x_i x_j x_k\left(\frac{5\beta+2}{2\beta+2}\frac{1}{r^5} - \frac{5\beta}{2\beta+2}\frac{1}{r^7}\right) \qquad (35)$$

where β is the viscosity ratio of the droplet and surrounding fluid (the value $\beta = 0$ corresponds to a gas bubble), $r = (x_1^2+x_2^2+x_3^2)^{1/2}$.

The symmetric tensor $\mathbf{E}$ can always be brought to the diagonal form with the components E_1, E_2, E_3 by

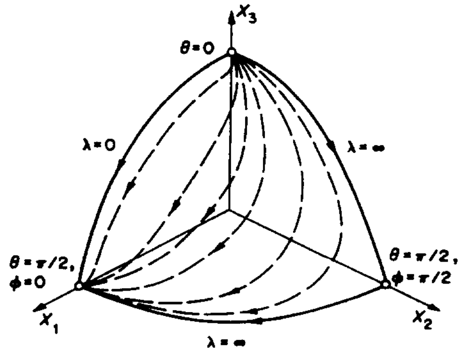

FIG. 2. Limiting streamlines on the surface of a spherical droplet in a three-dimensional deformational shear flow.

applying the proper rotation of the coordinate system. The diagonal elements $E_m (m = 1, 2, 3)$ determine the intensities of the stretching (compressing) motion along the principal axes of the tensor E. The values of E_m are the roots of the cubic equation

$$\det \| E_{ij} - \delta_{ij} E_m \| = 0.$$

The symmetric shear tensor has three scalar invariants

$$J_1 = E_{ij}\delta_{ij} = E_1 + E_2 + E_3 = 0$$

$$J_2 = (E_{ij}E_{ij})^{1/2} = (E_1^2 + E_2^2 + E_3^2)^{1/2}$$

$$J_3 = |\det \| E_{ij} \| |^{1/3} = |E_1 E_2 E_3|^{1/3}$$

which remain unchanged on any rotations (with reflections of the initial Cartesian coordinate system). Due to the fluid incompressibility $J_1 = 0$, only two out of three, diagonal components will be independent.

Further, the Cartesian coordinate system fixed with respect to the principal axes of the shear tensor is designated as X_1, X_2, X_3 (Fig. 2), and, for the sake of definiteness, it is considered that $E_1 \geqslant E_2 \geqslant 0, E_3 < 0$. In the spherical coordinate system r, θ, ϕ fixed with respect to the principal axes of the shear tensor the tangential fluid velocity vector components (35) on the droplet surface (at $r = 1$) have the form

$$v_\theta = \frac{\sin 2\theta}{4(\beta+1)}[-3E_3 + (E_1 - E_2)\cos 2\phi]$$

$$v_\phi = \frac{E_2 - E_1}{2(\beta+1)} \sin\theta \sin 2\phi. \qquad (36)$$

It is seen that the flow field is three-dimensional when $E_2 \neq E_1$.

It follows from equations (36) that on the spherical droplet surface there are six isolated singular critical points located on the principal axes of the shear tensor: (1) $\theta = 0$; (2) $\theta = \pi$; (3) $\theta = \pi/2, \phi = 0$; (4) $\theta = \pi/2, \phi = \pi$; (5) $\theta = \pi/2, \phi = \pi/2$; (6) $\theta = \pi/2, \phi = 3\pi/2$. The first two of which are the points of flow impingement, the following two are runoff points and the last two are neutral points (the saddle-point singularity). In the limiting axisymmetric case at $E_1 = E_2$ in lieu of the last four isolated critical points the critical runoff line appears at the droplet equator ($\theta = \pi/2$).

5.1. *The axisymmetric case*

First, consider the most simple axisymmetric case which corresponds to the values

$$E_1 = \tfrac{1}{2}, \quad E_2 = \tfrac{1}{2}, \quad E_3 = -1. \qquad (37)$$

Due to the symmetry of the problem about the plane $\theta = \pi/2$, it is sufficient to limit the discussion to the region $0 \leqslant \theta \leqslant \pi/2$. Assuming in equation (30) that $v_\phi = 0$ and taking into account equations (23) and (36), at $E_2 = E_1 = 1/2$, the function f can be defined as

$$f = v_\theta \sin\theta, \quad v_\theta = \frac{3}{2}\frac{\sin\theta\cos\theta}{\beta+1}. \qquad (38)$$

The velocity scale for dimensionless equations (38) was taken to be the quantity $U = a|E_3^*|$.

Evaluating the integrals

$$\int^\eta \frac{\sqrt{g^s}}{g^s_{\xi\xi}} f \,\mathrm{d}\eta = \int^\theta v_\theta \sin^2\theta \,\mathrm{d}\theta = \frac{3}{8}\frac{\sin^4\theta}{\beta+1}$$

$$\int^\eta \frac{\sqrt{g^s}}{f}\,\mathrm{d}\eta = \int^\theta \frac{\mathrm{d}\theta}{v_\theta} = \frac{2}{3}(\beta+1)\ln\tan\theta$$

and making use of equations (18) and (19), the new variable can be found

$$z = \frac{3}{8}\frac{1}{\beta+1}\left\{\sin^4\theta - \left[1 + \cot^2\theta \exp\left(\frac{3t}{\beta+1}\right)\right]^{-2}\right\}. \qquad (39)$$

Substituting equations (38) and (39) into equation (33b) gives the integral diffusion flux I per half the droplet surface (for $0 \leqslant \theta \leqslant \pi/2$). The doubling of this value will give the total diffusion flux per the entire droplet $I_\Sigma = 2I$. Allowing for the fact that the dimensionless spherical droplet surface is equal to 4π, the mean Sherwood number $Sh = I_\Sigma/(4\pi)$ can be defined as

$$Sh = \left\{\frac{3Pe}{2\pi(\beta+1)}\coth\left[\frac{3t}{2(\beta+1)}\right]\right\}^{1/2}$$

$$Pe = \frac{a^2|E_3^*|}{D}. \qquad (40)$$

It should be noted that equation (40) at $\beta = 0$ was obtained previously in ref. [4].

5.2. *The three-dimensional case*

Using equations (33a) the dependence of the mean Sherwood number $Sh = I_\Sigma/(4\pi)$ on time will be obtained for a droplet in an arbitrary deformational shear flow. To the three-dimensional flow field there corresponds the inequality $E_2 \neq E_1$ in the expressions for the components of fluid velocities, equations (36). It will be assumed, as before, that at first the concentration in the continuous phase is the same and that later the reaction starts to proceed diffusionally on the droplet surface.

All calculations will be conducted sequentially following the scheme given at the end of Section 4.

Characteristic equation (27), which determines the dependence of the curvilinear coordinate λ on the spherical coordinates θ, ϕ in view of equation (36), has the form

$$\frac{2\,\mathrm{d}\theta}{\sin 2\theta} = \frac{3E_3+(E_2-E_1)\cos 2\phi}{(E_1-E_2)\sin 2\phi}\,\mathrm{d}\phi. \tag{41}$$

For the present, consider the flow region in the first quadrant $0 \leqslant \theta, \phi \leqslant \pi/2$. The general solution to equation (41) can be presented in the form

$$\tan^2\theta\tan^\kappa\phi\sin 2\phi = A$$

where

$$\kappa = 3E_3/(E_2-E_1)$$

and A is an arbitrary constant. The dependence of the curvilinear coordinate λ on θ and ϕ is obtained by setting in the general solution that $A \equiv \lambda$

$$\lambda = \tan^2\theta\tan^\kappa\phi\sin 2\phi,\quad \kappa = 3\frac{E_1+E_2}{E_1-E_2}. \tag{42}$$

Here, for representing the exponent κ, the third component of the metric tensor was excluded with the aid of the equality $E_3 = -E_1-E_2$ (see the first equation in system (36)).

Figure 2 depicts the qualitative behaviour of the limiting streamlines on the droplet surface in the first quadrant $0 \leqslant \theta$, $\phi \leqslant \pi/2$; the value of the new variable, equation (42), varies within the range from zero to infinity.

Integrating characteristic equation (26) for parameters (37)

$$\frac{\cos\theta}{\sin\theta}\,\mathrm{d}\theta = \frac{\sin 2\phi\,\mathrm{d}\phi}{\kappa+\cos 2\phi},\quad \kappa = 3\frac{E_1+E_2}{E_1-E_2} \tag{43}$$

for another curvilinear coordinate η gives

$$\eta = \sin^2\theta(\kappa+\cos 2\phi). \tag{44}$$

The integrals will be calculated which enter into equation (18) for the time-similar variable z. To this end, the last equations in system (32) will be employed and the equality $\partial\lambda/\partial\theta = 4\lambda/\sin 2\theta$ resulting from equation (42) will be taken into consideration. This will yield

$$\int^\eta \frac{\sqrt{g^s}}{f}\,\mathrm{d}\eta = \frac{\beta+1}{E_1-E_2}\int^\phi \frac{2\,\mathrm{d}\phi}{\sin 2\phi} = \frac{\beta+1}{E_1-E_2}\ln\tan\phi$$

$$\int^\eta \frac{\sqrt{g^s}}{g^s_{\xi\xi}}f\,\mathrm{d}\eta = \frac{E_1-E_2}{32\lambda^2(\beta+1)}$$

$$\times\int^\phi \left\{\sin^2\theta\sin^2 2\theta\right\}_\lambda \sin 2\phi\,\mathrm{d}\phi = \frac{E_1-E_2}{32(\beta+1)}$$

$$\times\int^{\sin^2\phi}\frac{y^{(\kappa+1)/2}(1-y)^{\kappa-1}\,\mathrm{d}y}{\left[y^{(\kappa+1)/2}+\frac{\lambda}{2}(1-y)^{(\kappa-1)/2}\right]^3}. \tag{45}$$

The use of these expressions for the variable z, equation (18), gives

$$z = \frac{E_1-E_2}{32(\beta+1)}\int_{\mu(\omega)}^{\sin^2\phi} Q(y,\lambda)\,\mathrm{d}y$$

$$Q(y,\lambda) = \frac{y^{(\kappa+1)/2}(1-y)^{\kappa-1}}{\left[y^{(\kappa+1)/2}+\frac{\lambda}{2}(1-y)^{(\kappa-1)/2}\right]^3}$$

$$\mu(\omega) = \left\{1+\exp\left(2\frac{E_1-E_2}{\beta+1}\omega\right)\right\}^{-1}$$

$$= \left\{1+\cot^2\phi\exp\left(2\frac{E_1-E_2}{\beta+1}t\right)\right\}^{-1}. \tag{46}$$

To formulate equations (46), the notation $\mu(\omega) \equiv \sin^2 T(\omega)$ (see equations (18) and (19)) was employed and account was taken of

$$\omega = t-\frac{\beta+1}{E_1-E_2}\ln\tan\phi.$$

The mean Sherwood number $Sh = I_\Sigma/(4\pi)$ is calculated at $\Lambda = \infty$ from equation (21a) in the integrand of which the consequent transformations $\eta \to \phi-\sin^2\phi$ were performed with respect to the first variable of integration in view of equations (29)–(31) (in much the same way as the second integral in equation (45) is calculated). Use was also made of the fact that the integral diffusional flux to the part of the surface considered constitutes 1/8 of the total flux. The indicated procedure leads to the following expression for the mean Sherwood number:

$$Sh(t') = \frac{Pe_m^{1/2}}{(2\pi)^{3/2}}\left\{\frac{\sigma-1}{[2(1+\sigma+\sigma^2)]^{1/2}}\right\}^{1/2}$$

$$\times\int_0^\infty\int_0^1\frac{Q(x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda}{\sqrt{(\varepsilon(t',x,\lambda))}} \tag{47}$$

$$\varepsilon(t',x,\lambda) = \int_{p(x,t')}^x Q(y,\lambda)\,\mathrm{d}y$$

$$p(x,t') = \frac{x}{x+(1-x)\exp(2t')}$$

$$t' = \frac{E_1^*-E_2^*}{\beta+1}t_*,\quad Pe_m = \frac{a^2J_2^*}{D(\beta+1)}$$

$$\sigma = \frac{E_2^*}{E_1^*},\quad \kappa = 3\frac{1+\sigma}{1-\sigma}$$

$$J_2^* = (E_{ij}^*E_{ij}^*)^{1/2} = [2(E_1^{*2}+E_2^{*2}+E_1^*E_2^*)]^{1/2}.$$

Here, the superscript '*' corresponds to dimensional quantities, the function Q is determined in equation (46) and the modified Peclet number Pe_m is introduced

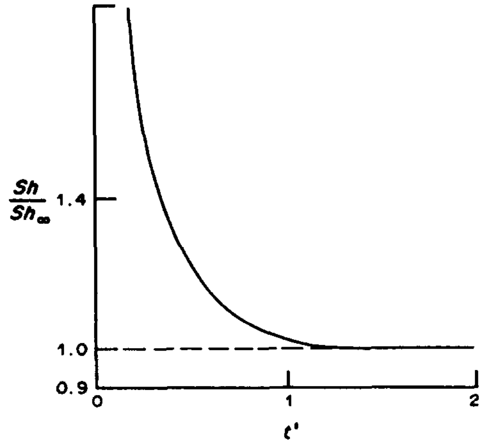

FIG. 3. The normalized mean Sherwood number vs dimensionless time for a spherical droplet in a plane shear flow.

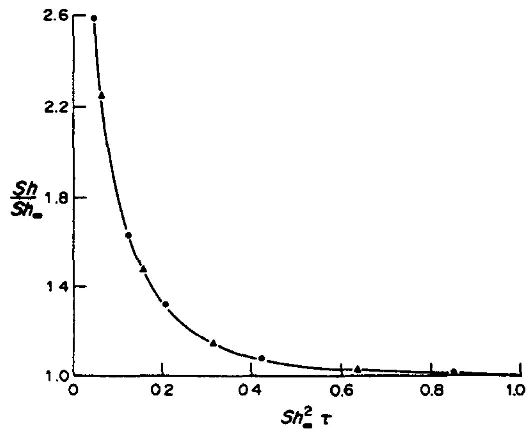

FIG. 4. Comparison of the predicted mean Sherwood number by equation (49) (——) with the data obtained by other authors: ●, refs. [1–3]; ▲, ref [12].

according to the second invariant of the matrix of shear coefficients just as it was done in refs. [5, 7, 8].

When $t' \to \infty$, $p \to 0$ and equations (47) go over into the results of ref. [8] which correspond to the established mass transfer regime.

Figure 3 shows the normalized mean Sherwood number Sh/Sh_∞ per spherical droplet in a plane shear flow at $E_2^* = 0$ (the respective flow field is three-dimensional, see equation (36)) calculated numerically based on formula (47). Here $Sh_\infty = 0.615\sqrt{(Pe_m)}$ is the mean Sherwood number corresponding to the steady mass exchange of the droplet with the plane shear flow [8].

6. AN APPROXIMATE EQUATION FOR THE MEAN SHERWOOD NUMBER IN THE CASE OF ARBITRARY FLOW ABOUT DROPLETS, BUBBLES AND SOLID PARTICLES

An approximate equation will be obtained to calculate the mean Sherwood number for unsteady mass exchange of droplets, bubbles and solid particles in a developed flow of arbitrary type at large Peclet numbers.

For this, it is necessary to proceed to the limit $t \to \infty$ in equation (40). As a result, the mean Sherwood number is obtained for steady mass exchange of a spherical droplet in an axisymmetric shear flow

$$Sh_\infty = \left[\frac{3Pe}{2\pi(\beta+1)}\right]^{1/2}. \tag{48}$$

Excluding the Peclet number from equations (40) and (48), equation (40) will be presented in the following equivalent form [10]:

$$\frac{Sh}{Sh_\infty} = \{\coth(\pi\, Sh_\infty^2\, \tau)\}^{1/2}, \quad \tau = \frac{Dt_*}{a^2} \tag{49}$$

where $\tau = t/Pe$ is the new dimensionless time.

Now, 'forgetting' that the coefficient Sh_∞ in equation (49) is specified by asymptotics of equation (40) when $t \to \infty$ and determining this coefficient directly from the solution of the corresponding stationary problem of convective mass and heat transfer at large Peclet numbers, equation (49) can be used with success for describing approximately the unsteady diffusion to the surfaces of droplets, particles and bubbles of any shape in an arbitrary flow.

The fitting of the approximate equation obtained by the substitution into equation (49) of the corresponding stationary value of [11]

$$Sh_\infty = \left[\frac{2Pe}{3\pi(\beta+1)}\right]^{1/2}, \quad Pe = \frac{aU_\infty}{D} \tag{50}$$

to the results of refs. [1–3] shows that in the case of unsteady mass exchange of a spherical droplet with the developed translational Stokes flow the maximum error of equations (49) and (50) is less than 1%. In Fig. 4 the curves corresponding to equations (49) and (50) and to the results of refs. [1–3] do not virtually differ (it should be remembered that the results of refs. [1–3] for the mean Sherwood number are expressed by a fairly complex integral which cannot be presented in a convenient analytical form of the kind of equation (49)).

Substituting into equation (49) the quantity [11]

$$Sh_\infty = 0.624Pe^{1/3}, \quad Pe = \frac{aU_\infty}{D} \tag{51}$$

gives an approximate equation for calculating the mean Sherwood number in the case of unsteady mass transfer to a solid spherical particle in a developed translational Stokes flow. The maximum difference between equations (49) and (50) and the approximation [12] to the numerical–analytical solution of a similar problem [13] constitutes less than 2% (see Fig. 4).

Recall that equation (49) is exact in the diffusion boundary layer approximation for unsteady diffusion

Table 1. Comparison of the mean Sherwood number predicted by equation (49) with the data of other authors for different cases of flow about spherical droplets, bubbles and solid particles

Type of particles	Kind of flow	Method of solution	Error (%)	Reference
Droplet, bubble	Axisymmetric shear flow	Analytical, DBLA†	0	[4]
Droplet, bubble	Translational Stokes flow	Analytical, DBLA	0.7	[1, 2]
Bubble	Laminar translational flow at large Reynolds numbers	Analytical, DBLA	0.7	[2, 3]
Particle	Translational flow of ideal inviscid fluid	Analytical, DBLA	0.7	[2, 3]
Solid particle	Translational Stokes flow	Series expansion + approximation	1.4	[12]
Droplet, bubble	Three-dimensional shear Stokes flow	Analytical, DBLA	1.8	Present work
Droplet, bubble	Flow formed due to the presence of electrical field	Analytical, DBLA	0	Present work and [14]
Solid particle	Translational Stokes flow	Finite-difference numerical method (at $Pe = 500$)	4	[15]

† DBLA, diffusion boundary layer approximation.

to a spherical droplet in an axisymmetric translational flow. Comparison with the predicted data presented in Figs. 3 and 4 shows that the error of the approximate equation (49) in the case of a plane shear (corresponds to the three-dimensional flow field, equation (36), at $E_2 = 0$) constitutes less than 1.8%.

In a similar fashion approximate equations can be also obtained for other non-stationary problems. For example, using the auxiliary quantity [7]

$$Sh_\infty = 0.9 Pe_m^{1/3}, \quad Pe_m = \frac{a^2 J_2^*}{D}, \quad J_2^* = (E_{ij}^* E_{ij}^*)^{1/2}$$

in equation (49) makes it possible to describe the process of the establishment of mass transfer to the surface of a spherical solid particle in an arbitrary deformational linear shear Stokes flow (in this case the fluid velocity components are determined by equations (35) at $\beta = \infty$). Moreover, the substitution into equation (49) of the auxiliary value of the mean Sherwood number $Sh_\infty = (2Pe/\pi)^{1/2}$, which corresponds to a steady process of diffusion to a sphere in a potential translational ideal (inviscid) fluid flow, leads to a very insignificant error (<1%) as compared with exact results [2, 3].

Table 1 (see also Fig. 4) presents the comparison of the mean Sherwood number predicted by equation (49) with the data of other authors obtained for different cases of flow around spherical droplets, bubbles and solid particles.

It should be noted that equation (49) can also be used to calculate unsteady mass transfer of droplets, bubbles and solid particles of non-spherical shape. In this case the mean Sherwood number should be defined as the ratio of the dimensionless total (integral) diffusional flux to the dimensionless particle surface area. Then the approximate equation (49) will ensure the correct asymptotic result at small and large values of the dimensionless time τ.

7. PROBLEMS OF STEADY-STATE MASS EXCHANGE BETWEEN DROPLETS, BUBBLES AND SOLID PARTICLES AND A FLOW IN THE PRESENCE OF VOLUMETRIC REACTION

Consider the developed mass exchange between a spherical particle (droplet) of radius a and a fluid flow when the substance diffusing from the particle surface undergoes the first-order chemical transformation with the rate $W = KC_*$ in the external phase volume where C_* is the concentration, K the constant of the volumetric chemical reaction rate.

In dimensionless variables in the spherical coordinate system r, θ, ϕ, fixed with respect to the particle, the process of mass transfer in the fluid is represented by the following equation and boundary conditions:

$$Pe(\mathbf{v}\nabla)\bar{c} = \Delta\bar{c} - k\bar{c} \tag{52}$$

$$r = 1, \quad \bar{c} = 1 \tag{53}$$

$$r \to \infty, \quad \bar{c} \to 0$$

$$\bar{c} = C_*/C_s, \quad Pe = aU/D, \quad k = a^2K/D \tag{54}$$

where C_s is the concentration on the particle surface, $\mathbf{v}$ the fluid velocity distribution considered to be known from the solution of the corresponding hydrodynamic problem.

For solving the stationary problems of convective mass exchange of droplets and particles with fluids in the course of the first-order volumetric chemical reaction, it is convenient to use the results obtained by solving the corresponding non-stationary problems without volumetric reaction. To see this, consider the equation

$$\frac{\partial c}{\partial \tau} + Pe(\mathbf{v}\nabla)c = \Delta c \tag{55}$$

under initial and boundary conditions

$$\tau = 0, c = 0: \quad r = 1, c = 1; \quad r \to \infty, c \to 0 \tag{56}$$

where $\mathbf{v}$ is the fluid velocity vector corresponding to the stationary flow field.

Applying the Laplace–Carson transformation (with the actual parameter k) to equations (55) and (56)

$$\bar{c} = k \int_0^\infty \exp(-k\tau) c \, d\tau \tag{57}$$

the stationary problem in the presence of the first-order volumetric chemical reaction, equations (52)–(54), is obtained.

It follows from equation (57) that the mean Sherwood number

$$\overline{Sh} = -\frac{1}{2} \int_0^\pi \sin\theta \left(\frac{\partial \bar{c}}{\partial r}\right)_{r=1} d\theta$$

which corresponds to the solution of problem (52)–(54), can be expressed in terms of the auxiliary Sherwood number Sh determined by solving non-stationary problem (55), (56) in the following fashion:

$$\overline{Sh} = k \int_0^\infty \exp(-k\tau) Sh \, d\tau. \tag{58}$$

A useful assessment will be obtained which will be needed in what follows. Assume that Sh is the mean Sherwood number corresponding to the exact solution of auxiliary problem (55), (56) and Sh_{ap} is the approximate expression for the Sherwood number the error of which is δ, i.e.

$$|Sh - Sh_{ap}| \leqslant \delta. \tag{59}$$

Applying the Laplace–Carson transformation to the difference $Sh - Sh_{ap}$ in view of equation (59) yields the inequality

$$\overline{Sh} - \overline{Sh}_{ap} = k \int_0^\infty \exp(-k\tau)(Sh - Sh_{ap}) \, d\tau$$

$$\leqslant \delta k \int_0^\infty \exp(-k\tau) \, d\tau = \delta \tag{60}$$

where $\overline{Sh}$ and $\overline{Sh}_{ap}$ are the exact and approximate values of the mean Sherwood number, respectively, which correspond to the solution of the stationary problem with the first-order volumetric chemical reaction, equations (52)–(54). Similarly it can be found that $\overline{Sh}_{ap} - \overline{Sh} \leqslant \delta$. Taking into account equation (60), this yields the inequality

$$|\overline{Sh} - \overline{Sh}_{ap}| \leqslant \delta. \tag{61}$$

The assessment (61) shows that, having obtained a sufficiently good approximate relation for the auxiliary Sherwood number in the non-stationary problem by applying the Laplace–Carson transformation, it is possible to get a good approximate expression (with the same precision) for the mean Sherwood number in

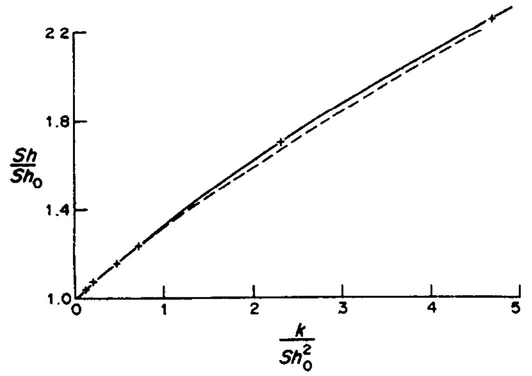

FIG. 5. The mean Sherwood number vs the dimensionless reaction rate constant: ——, calculation by equation (62); -----, calculation by equation (64); × × × × ×, ref. [16].

the stationary problem with the first-order volumetric chemical reaction.

Taking into account the aforegoing, use will now be made of the results of Section 6 in which the diffusion boundary layer non-stationary problems were considered.

Equation (49) will be employed as the auxiliary mean Sherwood number.

Applying the Laplace–Carson transformation to equation (49) an approximate solution will be obtained for a series of corresponding stationary equations, equations (52)–(54), with the first-order volumetric chemical reaction in the form

$$\frac{\overline{Sh}}{Sh_\infty} = F\left(\frac{k}{\overline{Sh}_0^2}\right) \tag{62}$$

where the function F is prescribed by the integral

$$F(x) = x \int_0^\infty \exp(-x\tau)[\coth(\pi\tau)]^{1/2} \, d\tau. \tag{63}$$

In equation (62) the quantity $\overline{Sh}_0$ corresponds to the mean Sherwood number in the absence of volumetric chemical reaction, i.e. at $k = 0$. (In writing equation (62) allowance has been made for the equality $Sh_\infty = \overline{Sh}_0$, where the quantity $Sh_\infty = \lim_{\tau \to \infty} Sh$ corresponds to the developed diffusion regime in non-stationary problem (55), (56) and the quantity $\overline{Sh}_0 = \lim_{k \to 0} \overline{Sh}$ determines the mean Sherwood number in problem (52)–(54) at $k = 0$.)

Approximate equation (62) adequately displays the structure of the dependence of the mean Sherwood number on the complex $k/\overline{Sh}_0^2$ at large Peclet numbers (in the diffusion boundary layer approximation). When $k \to 0$, equations (62) and (63) yield the exact result, i.e. $\overline{Sh} \to \overline{Sh}_0$. In the other limiting case of $k \to \infty$ (Pe = const.), the correct asymptotic result, i.e. that $\overline{Sh} \to \sqrt{k}$ [5], is obtained from equation (62). This equation also ensures an accurate asymptotic result when $Pe \to \infty$ (k = const.), since in this case $\overline{Sh}_0 \to \infty$ and $\overline{Sh} \to \overline{Sh}_0$.

Figure 5 presents the curves plotted from equations (62) and (63). The points correspond to the solution which was obtained for the problem of mass exchange of a spherical droplet with a translational Stokes flow by another technique in ref. [16] (see also ref. [6]).

It is important to observe that for the axisymmetric shear flow around a spherical droplet, equations (62) and (63) are exact. The maximum error of equation (62) for some other cases of mass exchange of droplets, bubbles and solid particles in different flows in the presence of the first-order volumetric chemical reaction can be estimated using Table 1, with the result of Section 6 taken into account. In particular, it follows from Table 1 that the solution of the three-dimensional problem on diffusion to a spherical droplet in a plane shear flow leads to the relation for the mean Sherwood number which differs from equations (62) and (63) by less than 1.8%.

For approximate calculations of the mean Sherwood number, it is possible to make use of the following simple expression:

$$\overline{Sh} = \sqrt{k} \coth (\sqrt{k/\overline{Sh}_0^2}) \tag{64}$$

which differs from more complex equations (62) and (63) by less than 2%. Equation (64) is shown in Fig. 5 by a dashed line.

It should also be noted that the root of the cubic equation

$$\overline{Sh}^3 - k\,\overline{Sh} - \overline{Sh}_0^3 = 0 \tag{65}$$

differs from equation (62) by less than 1% at most.

Thus, it is shown that equations (62), (64) and (65) can be successfully used for the approximate determination of the mean Sherwood number in problems on mass transfer of droplets, particles and bubbles in different types of flows in the presence of the first-order volumetric chemical reaction at large Peclet numbers. Recall that the parameter $\overline{Sh}_0$ corresponds to the mean Sherwood number in analogous, more simple problems at $k = 0$, i.e. in the absence of volumetric reaction.

Remark. For droplets and particles of non-spherical shape the mean Sherwood number in equations (63)–(65) is to be determined from the equation $\overline{Sh} = \bar{I}/S$, where $\bar{I}$ is the dimensionless integral (total) diffusion flux per particle, S the dimensionless particle surface area. In this case approximate equations (62), (64) and (65) will guarantee a correct asymptotic result in three limiting cases: (1) $k \to 0$ (Pe = const.); (2) $k \to \infty$ (Pe = const.); (3) $Pe \to \infty$ (k = const.).

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PROBLEMES TRIDIMENSIONNELS DE COUCHE LIMITE DE DIFFUSION VARIABLE

Résumé—Dans l'approximation de la couche limite de diffusion, on obtient une solution analytique exacte pour le problème de l'échange convectif variable de masse entre une gouttelette (bulle) sphérique et un écoulement cisaillant arbitraire, avec un champ de vitesse non perturbé décrit par un tenseur symétrique des déformations. On établit la dépendance du nombre de Sherwood vis-à-vis du temps. On présente une formule approchée simple pour le calcul de l'échange de masse pour des gouttes et des particules solides avec un écoulement arbitraire permanent. On considère le problème stationnaire d'échange de masse entre une gouttelette et un écoulement cisaillant linéaire, en présence d'une réaction chimique du premier ordre. On suggère une équation pour calculer le nombre de Sherwood pour une gouttelette ou une particule de forme arbitraire et pour un type quelconque d'écoulement à grand nombre de Péclet dans le domaine entier des constantes de vitesse de réaction.

DREIDIMENSIONALE GRENZSCHICHTEN MIT INSTATIONÄRER DIFFUSION

Zusammenfassung—Bei Näherungsbetrachtungen für Diffusionsgrenzschichten ergibt sich eine exakte analytische Lösung für das Problem des instationären konvektiven Stoffaustauschs zwischen einem kugeligen Tropfen (Blase) und einer beliebigen dreidimensionalen linearen Scherströmung. Das ungestörte Geschwindigkeitsfeld ist durch einen symmetrischen Schertensor gekennzeichnet. Es ergibt sich die Abhängigkeit der Sherwood-Zahl von der Zeit und der Peclet-Zahl. Eine einfache Näherungsgleichung für die Berechnung des instationären Stoffaustauschs von Tropfen und Feststoffpartikeln mit einer beliebigen stationären Strömung wird vorgestellt. Außerdem wird das stationäre Stoffübergangsproblem zwischen einem Tropfen und einer linearen Scherströmung in Anwesenheit einer volumetrischen chemischen Reaktion erster Ordnung betrachtet. Es wird eine Gleichung vorgeschlagen zur Berechnung der Sherwood-Zahl für einen Tropfen oder ein Partikel beliebiger Form und für jede Strömungsart (große Peclet-Zahlen) über den gesamten Bereich von Reaktionskonstanten.

ТРЕХМЕРНЫЕ ЗАДАЧИ НЕСТАЦИОНАРНОГО ДИФФУЗИОННОГО ПОГРАНИЧНОГО СЛОЯ

Аннотация—В приближении диффузионного пограничного слоя получено точное аналитическое решение задачи о нестационарном конвективном массообмене сферической капли (пузыря) с произвольным трехмерным линейным сдвиговым потоком, невозмущенное поле скоростей которого задается симметричным тензором сдвига. Определена зависимость числа Шервуда от времени и числа Пекле. Приведена простая приближенная формула для расчетаа интенсивности нестационарного массообмена капель и твердых частиц с произвольным установившимся потоком. Рассмотрена стационарная задача о массообмене капли в линейном сдвиговом потоке при протекании объемной химической реакции первого порядка. Предложена формула, позволяющая вычислять число Шервуда на каплю и частицу произвольной формы и любого типа течения при больших числах Пекле во всем диапазоне изменения константы скорости реакции.